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Destruction of higher-order squeezing by thermal noise

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Abstract. We examine the influence of thermal noise on several non-classical properties of squeezed number states. In order to evaluate arbitrary-order moments of both amplitude and quadrature operators of the superposition, we use a normal-ordering technique based on McCoy's theorem. Our analytical results are compact formulae involving Gauss hypergeometric functions.

We find that the thermal mean occupancy sufficient to destroy squeezing to any order is less than $\frac{1}{2}$. Other non-classical features of a squeezed number state, such as pairwise oscillations in the photon number distribution, sub-Poissonian statistics and amplitude-squared squeezing disappear completely above this threshold.

1. Introduction

One of the most striking problems concerning the non-classical properties of the electromagnetic field states is the extent to which they survive in the presence of noise and losses. For both squeezed states and quantum-mechanical superpositions of coherent states, the influence of losses [1–5] or thermal noise [6–9] on the oscillations in the photon-number distribution was extensively studied. It was found that small amounts of thermal noise destroy the oscillatory behaviour of the distribution. By interference-in-phase-space methods, a rather drastically restrictive condition on the thermal mean occupancy still maintaining the oscillations, has recently been given in the case of a squeezed coherent state (SCS) [9].

The aim of the present paper is to investigate the influence of thermal noise on the higher-order squeezing properties of a fundamental quantum state, the squeezed number state (SNS) [10–12]. The remarkable non-classical properties of this state, especially when analysing higher-order squeezing have recently received a good deal of attention [11, 12]. It was found that squeezing to different orders sets in at different values of the squeeze parameter. This fact indicates the existence of intrinsic higher-order squeezing, a concept introduced by Hong and Mandel [13]. The normally ordered moments of the quadrature operators were found to be oscillatory-in-sign functions with respect to the squeeze parameter [12]. Therefore, we analyse in the following the preservation of these properties in the presence of thermal noise. In section 2 we study the single-mode superposition of an arbitrary radiation field with a thermal one. General formulae are given here for both the density matrix and correlation functions. In section 3 we specialize them to the case of a single-mode field in a SNS. The resulting field is in a mixed state, hereafter called *thermal squeezed number state* (TSNS). We give first the density matrix of this superposition in the Fock basis.

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Plots of the photon number distribution for several values of thermal mean occupancy are presented. Then, by direct differentiation of the normally ordered CF of a TSNS, we obtain the correlation functions of arbitrary orders. In particular, we examine the second-order degree of coherence and establish the influence of the thermal component of the field on the sub-Poissonian photon statistics.

In section 4 we investigate the way thermal noise modifies the conditions for higher-order squeezing. For the sake of completeness, we examine both quadrature squeezing as defined by Hong and Mandel [13] and amplitude-squared squeezing as introduced by Hillery [14].

Section 5 is a summary of our physical results. We stress also that our analytical calculations are performed using a normal ordering technique to obtain the higher-order moments of both amplitude and quadrature operators in terms of Gauss hypergeometric functions. In the appendix we recall some useful properties of the hypergeometric functions ${}_1F_1$ and ${}_2F_1$. A remarkable summation formula is employed to get the main result in section 4.

2. Superposition of fields

In the present work we deal with a single-mode radiation field whose amplitude operators are denoted by a and $a^\dagger$. For any state of the field the CF [15] defined as the expectation value of the Weyl's displacement operator $D(\lambda) = \exp(\lambda a^\dagger - \lambda^* a)$,

$$\chi(\lambda) = \mathrm{Tr}[\rho D(\lambda)] \tag{2.1}$$

determines uniquely the density operator ρ. Indeed, the CF is the weight function in the Weyl expansion [16] of the density operator,

$$\rho = \frac{1}{\pi}\int \mathrm{d}^2\lambda\, \chi(\lambda) D(-\lambda). \tag{2.2}$$

The normally ordered CF,

$$\chi^{(\mathcal{N})}(\lambda) = \mathrm{Tr}[\rho \mathrm{e}^{\lambda a^\dagger} \mathrm{e}^{-\lambda^* a}] \tag{2.3}$$

has the Taylor expansion

$$\chi^{(\mathcal{N})}(\lambda) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{l!m!}\lambda^l(-\lambda^*)^m \langle (a^\dagger)^l a^m\rangle \tag{2.4}$$

from which one readily finds the correlation functions $\langle (a^\dagger)^l a^m\rangle$. Its Fourier transform is the well known Glauber's P-representation,

$$P(\beta) = \frac{1}{\pi}\int \mathrm{d}^2\lambda\, \exp(\beta\lambda^* - \beta^*\lambda)\chi_N(\lambda). \tag{2.5}$$

Now, following Glauber [17], we define the superposition of the one-mode field described by the density operator ρ_1 with a thermal noise in the same mode. The density operator of the superposition is

$$\rho = \int \mathrm{d}^2\beta\, P_T(\beta) D(\beta)\rho_1 D^\dagger(\beta) \tag{2.6}$$

where

$$P_T(\beta) = \frac{1}{\pi\bar{n}} \exp\left(-\frac{|\beta|^2}{\bar{n}}\right) \tag{2.7}$$

is Glauber's P-representation for the thermal field with the mean occupancy $\bar{n}$. equation (2.6) can be used conveniently to get the density operator of the superposition in the Fock basis. To this end we need the matrix elements of the Weyl's displacement operator [18, 19]

$$\langle k|D(\beta)|l\rangle = \left(\frac{l!}{k!}\right)^{1/2} \beta^{k-l} \exp\left(-\frac{|\beta|^2}{2}\right) L_l^{(k-l)}(|\beta|^2) \tag{2.8}$$

where $L_l^{(k-l)}$ is a Laguerre polynomial, equation (A3). We employ the polar coordinates in the integral (2.6) and are left with the relation

$$\langle m|\rho|n\rangle = \frac{1}{\bar{n}} \sum_{k,l} \left[\frac{m!n!}{k!l!}\right]^{1/2} \frac{1}{[(m-k)!]^2} \langle k|\rho_1|l\rangle \delta_{m-k,n-l} \int_0^\infty \mathrm{d}t\ \exp\left[-\left(1+\frac{1}{\bar{n}}\right)t\right] t^{m-k}$$
$$\times\, {}_1F_1(-k; m-k+1; t)\, {}_1F_1(-l; m-k+1; t). \tag{2.9}$$

The remaining integral is of the type (A4). After performing it we arrive at the summation

$$\langle m|\rho|n\rangle = \frac{1}{(m-n)!} \left[\frac{m!}{n!}\right]^{1/2} \frac{\bar{n}^n}{(\bar{n}+1)^m} \sum_{p=0}^{\infty} \left[\frac{(m-n+p)!}{p!}\right]^{1/2} \left(\frac{\bar{n}}{\bar{n}+1}\right)^p$$
$$\times\, {}_2F_1\left(-p, -n; m-n+1; \frac{1}{\bar{n}^2}\right) \langle m-n+p|\rho_1|p\rangle \qquad m > n \tag{2.10}$$

in which ${}_2F_1$ stands for a Gauss hypergeometric function, equation (A1).

According to equation (2.4) the correlation functions $\langle (a^\dagger)^l a^m\rangle$ of the superposition can be computed by using the normally ordered CF. To this aim we insert the Weyl expansion (2.2) of the operator ρ_1 in equation (2.6) and then use the multiplication law of the Heisenberg–Weyl group,

$$D(\alpha)D(\beta) = \exp[\tfrac{1}{2}(\alpha\beta^* - \alpha^*\beta)]D(\alpha+\beta). \tag{2.11}$$

We get

$$\rho = \frac{1}{\pi} \int \mathrm{d}^2\lambda\, \chi_1(\lambda) D(-\lambda) \int \mathrm{d}^2\beta\, P_T(\beta) \exp(\beta^*\lambda - \beta\lambda^*). \tag{2.12}$$

From the Fourier inversion theorem of equation (2.5) applied in the second integral (2.12) we arrive, by comparison with equation (2.2), at the CF of the superposition

$$\chi^{(\mathcal{N})}(\lambda) = \chi_1^{(\mathcal{N})}(\lambda)\chi_T^{(\mathcal{N})}(\lambda), \tag{2.13}$$

where

$$\chi_T^{(\mathcal{N})}(\lambda) = \exp(-\bar{n}|\lambda|^2) \tag{2.14}$$

is the normally ordered CF for a thermal field. Therefore,

$$\langle (a^\dagger)^l a^m\rangle_T = \delta_{lm}\bar{n}^l l!. \tag{2.15}$$

From equations (2.13) and (2.15) we get the correlation functions of the superposition

$$\langle (a^\dagger)^l a^m\rangle = \sum_{p=0}^{min\{l,m\}} \binom{l}{p}\binom{m}{p} p!\bar{n}^p \langle (a^\dagger)^{l-p} a^{m-p}\rangle_1. \tag{2.16}$$

Note that the diagonal case $l = m$ in equation (2.16) was given in [20]. From equation (2.16) we also infer, by setting $l = 0$, that thermal noise has no influence on the expectation values of arbitrary powers of the field amplitude operators,

$$\langle a^k\rangle = \langle a^k\rangle_1. \tag{2.17}$$

3. Thermal squeezed number state

In the following, we study the admixture of thermal noise with a field in a SNS. We recall that SNS is a pure state obtained by the action of the squeeze operator,

$$S(\zeta) = \exp\{\tfrac{1}{2}[\zeta(a^\dagger)^2 - \zeta^* a^2]\} \tag{3.1}$$

on a state with a definite number of photons, $|M\rangle$,

$$|M\rangle_S = S(\zeta)|M\rangle. \tag{3.2}$$

In equation (2.14) ζ is the complex squeeze parameter,

$$\zeta = r\exp(\mathrm{i}\phi) \qquad (r \geqslant 0,\ -\pi < \phi \leqslant \pi). \tag{3.3}$$

The CF associated to an SNS,

$$\chi_{SN}(\lambda) = \langle M|S^\dagger(\zeta)D(\lambda)S(\zeta)|M\rangle \tag{3.4}$$

is readily calculated by using the identity [21]

$$S^\dagger(\zeta)D(\lambda)S(\zeta) = D(\lambda\cosh r - \lambda^* \mathrm{e}^{\mathrm{i}\phi}\sinh r) \tag{3.5}$$

and then the diagonal case of equation (2.8). We get

$$\chi_{SN}(\lambda) = \exp[-\tfrac{1}{2}|\xi|^2]L_M^{(0)}(|\xi|^2). \tag{3.6}$$

In equation (3.6) we have denoted

$$\xi = \lambda\cosh r - \lambda^* \exp(\mathrm{i}\phi)\sinh r \tag{3.7}$$

while $L_M^{(0)}$ is a Laguerre polynomial. According to equation (2.13), the superposition TSNS has the normally ordered CF

$$\chi_{TSN}^{(\mathcal{N})}(\lambda) = \exp[-(\bar{n} - \tfrac{1}{2})|\lambda|^2 - \tfrac{1}{2}|\xi|^2]L_M^{(0)}(|\xi|^2). \tag{3.8}$$

3.1. Correlation functions

For the sake of simplicity, we prefer getting the correlation functions by direct differentiation of equation (3.8) rather than using equation (2.16). Accordingly, we make use of the differential operator

$$\frac{\partial}{\partial\lambda} = \cosh r\frac{\partial}{\partial\xi} + \exp(-\mathrm{i}\phi)\sinh r\left(-\frac{\partial}{\partial\xi^*}\right) \tag{3.9}$$

and its conjugate. As the Laguerre polynomial is proportional to a confluent hypergeometric function ${}_1F_1$, equation (A3), we can use the Humbert's relation (A5) to write simply

$$\exp[-\tfrac{1}{2}|\xi|^2]L_M^{(0)}(|\xi|^2) = \sum_{k=0}^{\infty}\frac{1}{k!}(-\xi\xi^*)^k\,{}_2F_1(-k, -M; 1; 2). \tag{3.10}$$

Due to the special form of the right-hand side of equation (3.10), the derivatives of equation (3.8) with respect to λ and λ^* can be performed to any order via equation (3.9). After some algebra we are able to get the expressions

$$\begin{aligned}\langle(a^\dagger)^l a^m\rangle_{TSN} &= \exp[-\mathrm{i}(l-m)\phi/2](\tfrac{1}{2})^{(l+m)/2}\frac{l!m!}{(|l-m|/2)!}(\tanh r)^{|l-m|/2}(\cosh r)^{l+m}\\ &\quad\times\sum_p\frac{1}{p![(l+m-|l-m|)/2-p]!}\left[\frac{2\bar{n}-1}{(\cosh r)^2}\right]^p\\ &\quad\times{}_2F_1(-\tfrac{1}{2}(l+m)+p, -M; 1; 2)\end{aligned}$$

$$\times {}_2F_1(-\tfrac{1}{2}(l+m)+p, -\tfrac{1}{2}(l+m-|l-m|)+p; \tfrac{1}{2}|l-m|+1; (\tanh r)^2)$$
$$(l+m) \text{ even} \qquad (3.11a)$$

$$\langle (a^\dagger)^l a^m \rangle_{TSN} = 0 \qquad (l+m) \text{ odd.} \qquad (3.11b)$$

For $l = 0$, equations (3.11) reduce, in agreement to equation (2.17), to the formula

$$\langle a^m \rangle_{TSN} = \begin{cases} (m-1)!![\exp(\mathrm{i}\phi)\sinh r \cosh r]^{m/2} {}_2F_1\left(-M, -\frac{m}{2}; 1; 2\right) & m \text{ even} \\ 0 & m \text{ odd} \end{cases} \qquad (3.12)$$

which is characteristic for a SNS [12]. In the diagonal case, equations (3.11) give the factorial moments of the photon number distribution,

$$\langle (a^\dagger)^l a^l \rangle_{TSN} = (\bar{n} - 12)^l l! \sum_{p=0}^{l} \binom{l}{p} (\cosh r)^{2p} (2\bar{n}-1)^{-p} {}_2F_1(-M, -p; 1; 2)$$
$$\times {}_2F_1(-p, -p; 1; (\tanh r)^2). \qquad (3.13)$$

Notice the important particular cases $l = 1$ and $l = 2$ in equation (3.13):

$$\langle a^\dagger a \rangle_{TSN} = \bar{n} + M + (2M+1)(\sinh r)^2 \qquad (3.14)$$

and

$$\langle (a^\dagger)^2 a^2 \rangle_{TSN} = 2\bar{n}(\bar{n}+2M) + M(M-1) + [6M^2 + 2M + 1 + 4\bar{n}(2M+1)](\sinh r)^2$$
$$+3(2M^2 + 2M + 1)(\sinh r)^4. \qquad (3.15)$$

We evaluate now the second-order degree of coherence

$$g^{(2)}_{TSN}(0) \equiv \frac{\langle (a^\dagger)^2 a^2 \rangle}{\langle a^\dagger a \rangle^2} = 1 + \frac{1}{\{\langle (a^\dagger a \rangle_{TSN}\}^2}$$
$$\times \{-M(1-2\bar{n}) + \bar{n}^2 + [2\bar{n}(2M+1) + 2M^2 + 1](\sinh r)^2$$
$$+2(M^2 + M + 1)(\sinh r)^4\}. \qquad (3.16)$$

For weak squeezing and $M > 0$, a sub-Poissonian statistics is found in the absence of thermal noise [12]. From equation (3.16), we see that in the case of a TSNS, the statistics cannot be sub-Poissonian for $\bar{n} > \frac{1}{2}$. Therefore, *a thermal mean occupancy equal to* $\frac{1}{2}$ *is sufficient to destroy the antibunching effect of an SNS.*

3.2. Density matrix

It is instructive to present now the density matrix of a TSNS in the Fock basis. The density matrix of a SNS was written compactly by one of us in [12] by means of the matrix elements of the squeeze operator. We found that only even–even and odd–odd $\langle n|S(\zeta)|p\rangle$ are non-vanishing. Explicitly, for even n and p we have [12]

$$\langle n|S(\zeta)|p\rangle = \frac{(-1)^{p/2}[n!p!]^{1/2}}{\left(\frac{n}{2}\right)!\left(\frac{p}{2}\right)!} \exp\left[\mathrm{i}\frac{(n-p)}{2}\phi\right]\left(\frac{1}{2}\tanh r\right)^{(n+p)/2}$$
$$\times {}_2F_1\left(-\frac{n}{2}, -\frac{p}{2}; \frac{1}{2}; -\frac{1}{(\sinh r)^2}\right). \qquad (3.17)$$

We focus only on even M, the situation being quite similar in the case of odd M. By introducing the matrix element (3.17) in the general formula (2.10) we get

$$\langle m|\rho|n\rangle = \left(\frac{m!}{n!}\right)^{1/2}\frac{1}{(m-n)!}\frac{M!}{[(\frac{M}{2})!]^2}\exp\left[\mathrm{i}\frac{(m-n)}{2}\phi\right]\frac{\bar{n}^n}{(\bar{n}+1)^m}\frac{1}{\cosh r}\left[\frac{1}{2}\tanh r\right]^{\frac{(m-n)}{2}+M}$$
$$\times\sum_{p=0}^{\infty}\frac{(m-n+2p)!}{p![(m-n)/2+p]!}\left(\frac{\bar{n}}{\bar{n}+1}\frac{1}{2}\tanh r\right)^{2p}$$
$$\times\,{}_2F_1\left(-2p,-n;m-n+1;\frac{1}{\bar{n}^2}\right){}_2F_1\left(-p,-\frac{M}{2};\frac{1}{2};-\frac{1}{(\sinh r)^2}\right)$$
$$\times\,{}_2F_1\left(-\frac{m-n}{2}-p,-\frac{M}{2};\frac{1}{2};-\frac{1}{(\sinh r)^2}\right)\qquad (m-n)\ \text{even} \tag{3.18a}$$
$$\langle m|\rho|n\rangle = 0 \qquad (m-n)\ \text{odd}. \tag{3.18b}$$

Figures 1 and 2 illustrate the photon number distribution (case $m = n$ in equation (3.18a)). We have taken several values of the thermal mean occupancy lied between 0 and 1. In figure 1, $M = 2$ and the squeeze parameter is $r = 3$, while in figure 2 we have used $M = 4$ and $r = 1$. Obviously, the oscillatory behaviour is smoothed out by thermal noise.

4. Higher-order squeezing

In [12] one of us calculated in a simple form the conditions for both conventional and intrinsic arbitrary order squeezing for a SNS. Our aim now is to find similar analytical expressions in the case of a TSNS. We recall that, according to Hong and Mandel [13], a quantum state possesses Nth-order squeezing if

$$\langle(\Delta X_j)^N\rangle < 2^{-N}(N-1)!! \qquad N\ \text{even} \tag{4.1a}$$

and intrinsic higher-order squeezing if the normally ordered moments are negative,

$$\langle:(\Delta X_j)^N:\rangle < 0 \qquad N\ \text{even}. \tag{4.1b}$$

In equation (4.1), $\Delta X_j = X_j - \langle X_j\rangle$ and X_j is one of the quadrature operators defined as

$$X_1 = \tfrac{1}{2}(a+a^\dagger) \qquad X_2 = \frac{1}{2\mathrm{i}}(a-a^\dagger) \tag{4.2}$$

and satisfying the commutation relation

$$[X_1, X_2] = \frac{\mathrm{i}}{2}I. \tag{4.3}$$

On account of equation (3.12), the higher-order squeezing condition (4.1a) reads

$$\langle(a+a^\dagger)^N\rangle < (N-1)!! \qquad N\ \text{even}. \tag{4.4}$$

In order to calculate the left-hand side of this inequality one may use a normal-ordering formula given as equation (10.43) in the work [22] of Wilcox,

$$\langle(a+a^\dagger)^N\rangle = \sum_{k=0}^{[N/2]}\sum_{s=0}^{N-2k}\left(\frac{1}{2}\right)^k\frac{N!\langle(a^\dagger)^s a^{N-2k-s}\rangle}{k!s!(N-2k-s)!}. \tag{4.5}$$

Taking advantage of deriving in section 3 the correlation functions (3.11), we are able to write equation (4.5) as a finite triple summation which contains also a product of two Gauss

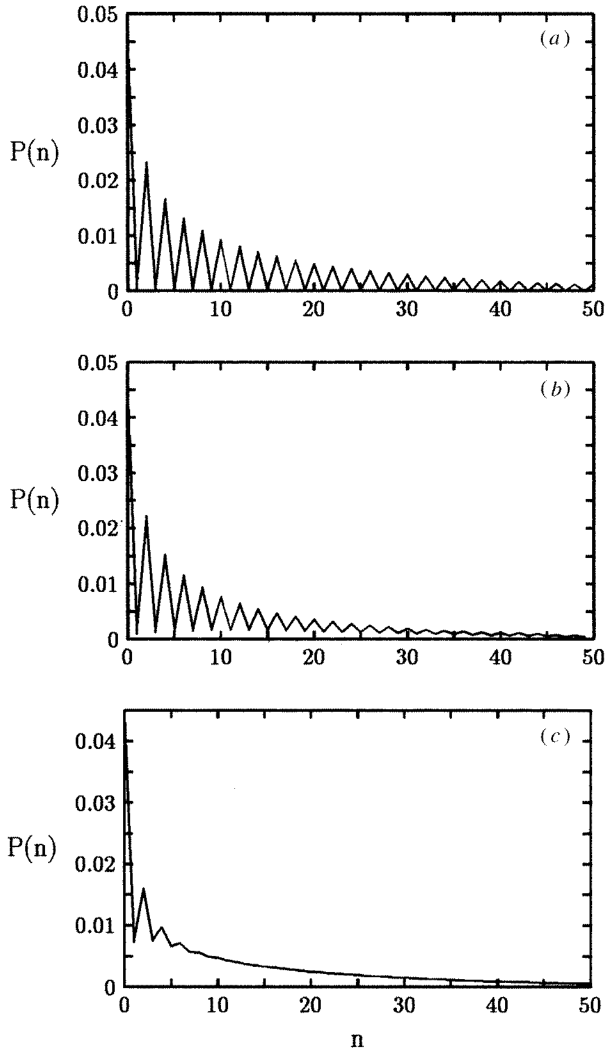

Figure 1. Photon number distribution for a TSNS with parameters $M = 2$ and $r = 3$. The values of the thermal mean occupancy are: (a) $\bar{n} = 0$, (b) $\bar{n} = 0.01$ and (c) $\bar{n} = 0.1$.

hypergeometric functions:

$$
\begin{aligned}
\langle (X_1)^N \rangle_{TSN} = {} & N!(\tfrac{1}{2})^{N/2} \sum_{k=0}^{N/2} \sum_{s=0}^{N-2k} \exp[-\mathrm{i}(N-2k-2s)\phi/2] \frac{1}{k!(N/2-k-s)!} \\
& \times (\tanh r)^{(N-2k-2s)/2} (\cosh r)^{N-2k} \sum_{p=0}^{s} \frac{(2\bar{n}-1)^p}{p!(s-p)!} (\cosh r)^{-2p} \\
& \times {}_2F_1\left(-\frac{N}{2}+k+p, -M; 1; 2\right) \\
& \times {}_2F_1\left(-\frac{N}{2}+k+p, -s+p; \frac{N}{2}-k-s+1; (\tanh r)^2\right).
\end{aligned} \tag{4.6}
$$

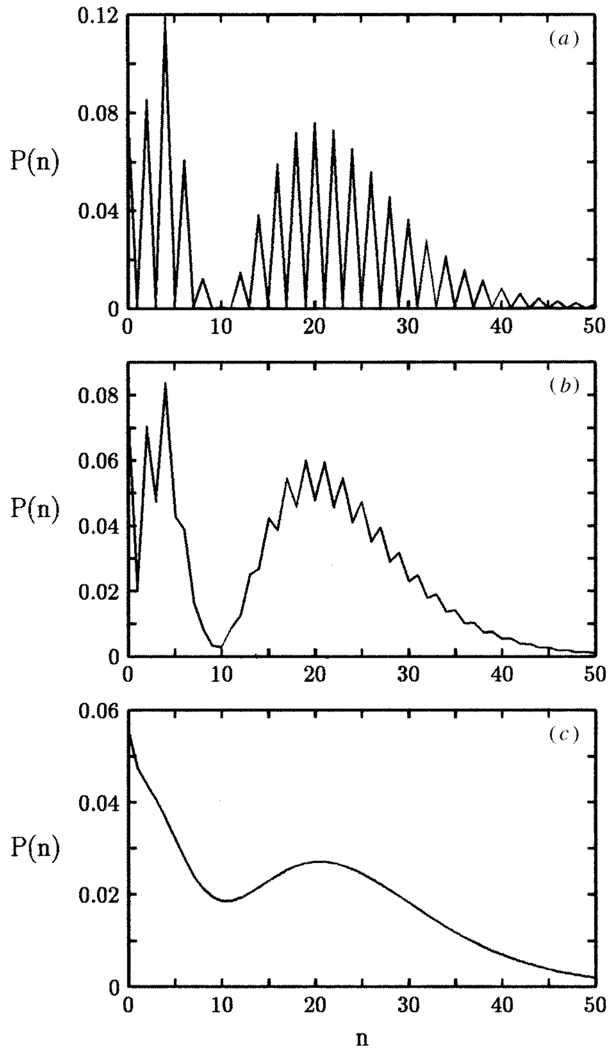

Figure 2. Photon number distribution for a TSNS with parameters $M = 4$ and $r = 1$. The values of the thermal mean occupancy are: (a) $\bar{n} = 0$, (b) $\bar{n} = 0.05$ and (c) $\bar{n} = 1$.

In the appendix, we show how this summation can be performed analytically to arrive at the remarkably simple formula

$$\langle (X_1)^N \rangle_{TSN} = 2^{-N}(N-1)!![|\alpha|^2 + 2\bar{n}]^{N/2}\, {}_2F_1\left(-M, -\frac{N}{2}; 1; \frac{2|\alpha|^2}{|\alpha|^2 + 2\bar{n}}\right) \tag{4.7}$$

where the parameter

$$\alpha = \cosh r + \mathrm{e}^{\mathrm{i}\phi} \sinh r \tag{4.8}$$

was introduced in [12]. When specializing $\bar{n} = 0$ in equation (4.7), we retrieve the result (26) from [12].

Using McCoy's theorem written in [22] as equation (5.6), we get the normally ordered moments as

$$\langle :(\Delta X_1)^N: \rangle = N! \sum_{l=0}^{N/2} \frac{(-1)^l \langle (\Delta X_1)^{N-2l} \rangle}{l!(N-2l)! 2^{3l}}. \tag{4.9}$$

After introducing in equation (4.9) the preceding result (4.7), we obtain a sum involving Gauss hypergeometric polynomials, which is of the type (A6). Actually, the normally ordered moments are

$$\langle :(\Delta X_1)^N: \rangle_{TSN} = 2^{-N}(N-1)!!(|\alpha|^2 + 2\bar{n} - 1)^{N/2} {}_2F_1\left(-M, -\frac{N}{2}; 1; \frac{2|\alpha|^2}{|\alpha|^2 + 2\bar{n} - 1}\right). \tag{4.10}$$

From equation (4.10) we learn that the condition for intrinsic squeezing cannot be fulfilled if $|\alpha|^2 + 2\bar{n} - 1$, and therefore the Gauss function ${}_2F_1$ are positive. For $\phi = \pi$, $|\alpha|^2 = \exp(-2r)$ so that we find the conditions necessary for the existence of intrinsic squeezing,

$$\bar{n} < \tfrac{1}{2} \qquad \text{and} \qquad r > r_s = -\tfrac{1}{2}\ln(1 - 2\bar{n}). \tag{4.11}$$

We also examine the strong squeezing limit $r \to \infty$ in equations (4.7) and (4.10):

$$\lim_{r\to\infty} \langle (\Delta X_1)^N \rangle_{TSN} = 2^{-N}(N-1)!!(2\bar{n})^{\frac{N}{2}} \tag{4.12}$$

$$\lim_{r\to\infty} \langle :(\Delta X_1)^N: \rangle_{TSN} = (-1)^{\frac{N}{2}} 2^{-N}(N-1)!!(1-2\bar{n})^{\frac{N}{2}}. \tag{4.13}$$

In this case, which is the most favourable to the existence of non-classical properties, equation (4.12) exhibits squeezing if and only if $\bar{n} < \frac{1}{2}$. Therefore, for strong squeezing this inequality is a necessary and *sufficient* condition for squeezing. In addition, according to equation (4.13), a TSNS is intrinsically squeezed for $\bar{n} < \frac{1}{2}$ only if $N/2$ is odd.

We stress now that the remarkable features of squeezing properties in the case of a SNS, as were found in [12], still survive in the presence of thermal noise if the conditions (4.11) are satisfied. Indeed, as shown in figures 3 and 4, the normally ordered moments $\langle :(\Delta X_1)^N: \rangle_{TSN}$ are oscillatory functions with respect to the squeeze parameter, but the oscillations are smoothed out when thermal mean occupancy goes towards the threshold value $\frac{1}{2}$. In figure 3 the dependence on r of fourth-order moments are given when $\bar{n}$ grows from 0.01 (figure 3(*a*)) to 0.45 (figure 3(*d*)). Figure 4 presents the same dependence, but for $N = 10$. At the same time, we point out that squeezing to different orders sets in at different values of the squeeze parameter $r_{min}^{(N)} > r_s$, which are dependent on the thermal mean occupancy. In tables 1 and 2 we present the onset values of the squeeze parameter for different M and several values of $\bar{n}$. In table 1, we give results for second-order squeezing, while in table 2 fourth-order squeezing is investigated.

A foregone conclusion to the above discussion is that other forms of higher-order squeezing could be influenced by thermal noise. We think at the amplitude-squared squeezing, a concept introduced by Hillery [14] in terms of the operators,

$$Y_1 = \tfrac{1}{2}(a^2 + (a^\dagger)^2) \qquad \text{and} \qquad Y_2 = \frac{1}{2\mathrm{i}}(a^2 - (a^\dagger)^2) \tag{4.14}$$

which correspond to the real and imaginary parts of the square of the field mode amplitude. The condition for squeezing in the Y_1 operator,

$$\langle (\Delta Y_1)^2 >\rangle < \langle a^\dagger a \rangle + \tfrac{1}{2} \tag{4.15}$$

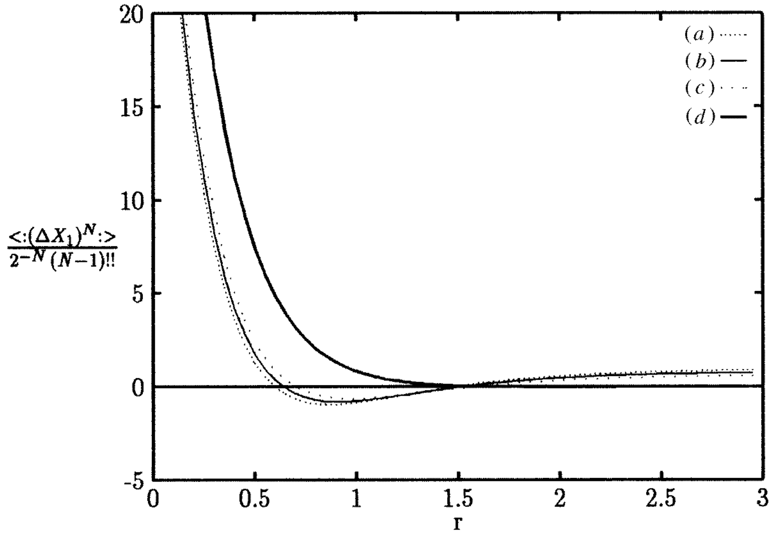

Figure 3. Normally ordered moment of fourth order for the quadrature X_1 in the case $M = 5$ versus the squeeze parameter for several values of thermal mean occupancy: (*a*) $\bar{n} = 0.01$, (*b*) $\bar{n} = 0.05$, (*c*) $\bar{n} = 0.10$ and (*d*) $\bar{n} = 0.45$.

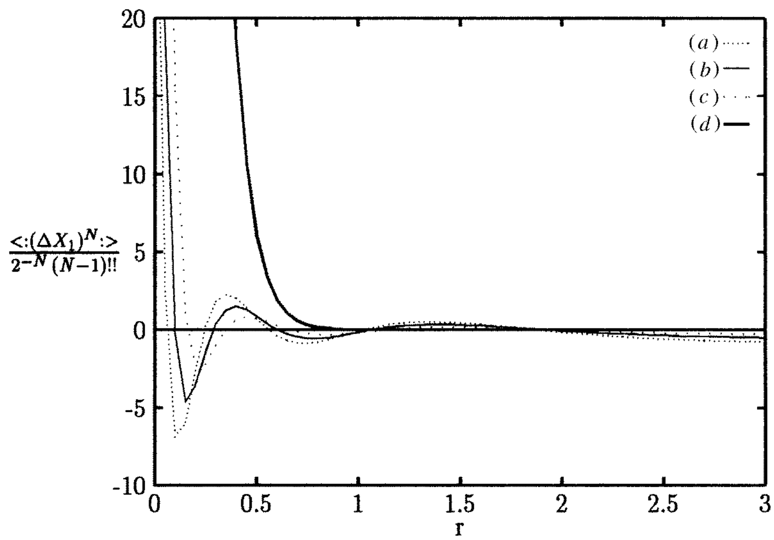

Figure 4. As in figure 1, but for the order $N = 10$.

originates in the commutation relation

$$[Y_1, Y_2] = \mathrm{i}(2a^\dagger a + I). \tag{4.16}$$

The squeezing defined by equation (4.15) involves fourth-order powers of the amplitude operators. Explicitly,

$$\langle(\Delta Y_1)^2\rangle = \langle a^\dagger a\rangle + \tfrac{1}{2} + \tfrac{1}{4}[\langle(\Delta(a^\dagger)^2)^2\rangle + \langle(\Delta a^2)^2\rangle] + \tfrac{1}{2}[\langle(a^\dagger)^2 a^2\rangle - \langle(a^\dagger)^2\rangle\langle a^2\rangle]. \tag{4.17}$$

Table 1. Onset values of the squeeze parameter for second-order squeezing of TSNS's with different M and $\bar{n}$.

	M			
$\bar{n}$	1	2	5	10
0	0.549	0.805	1.199	1.522
0.01	0.559	0.815	1.209	1.533
0.05	0.601	0.857	1.251	1.574
0.10	0.660	0.916	1.310	1.633
0.45	1.700	1.956	2.350	2.673

Table 2. As in table 1 but for fourth-order squeezing.

	M			
$\bar{n}$	1	2	5	10
0	0.402	0.641	1.027	1.349
0.01	0.416	0.655	1.042	1.363
0.05	0.472	0.713	1.101	1.423
0.10	0.547	0.791	1.179	1.502
0.45	1.689	1.943	2.337	2.661

By inserting equations (3.12) and (3.15) for a TSNS we obtain

$$\begin{aligned}\langle(\Delta Y_1)^2\rangle_{TSN} - \langle a^\dagger a\rangle_{TSN} - \tfrac{1}{2} = {} & \tfrac{1}{2}M(M-1) + \bar{n}(\bar{n}+2M) \\ & +[M(M-1)+2\bar{n}(2M+1)+(M^2+M+1)\cos(2\phi)](\sinh r)^2 \\ & +(M^2+M+1)(1+\cos(2\phi))(\sinh r)^4. \end{aligned} \quad (4.18)$$

The condition (4.15) could be satisfied only if $\cos(2\phi) < 0$. The maximum squeezing is reached for $\phi = \pi/2$, when

$$\langle(\Delta Y_1)^2\rangle_{TSN} - \langle a^\dagger a\rangle_{TSN} - \tfrac{1}{2} = \tfrac{1}{2}M(M-1) + \bar{n}(\bar{n}+2M) + (2\bar{n}-1)(2M+1)(\sinh r)^2. \quad (4.19)$$

The right-hand side of equation (4.19) could be negative only if $\bar{n} < \frac{1}{2}$.

Now, we are in a position to draw the final conclusion of our analysis:*the value $\bar{n} = \frac{1}{2}$ of the thermal mean occupancy is sufficient to destroy usual squeezing to any order and also amplitude-squared squeezing*.

5. Summary

In this paper we have analysed the preservation of some non-classical properties of a SNS field when superposed on a thermal field. We have found that the conditions for both conventional and intrinsic higher-order squeezing, as introduced by Hong and Mandel, are expressed by formulae similar to those describing the pure field in a SNS. However, the squeezing properties are altered by admixture with thermal noise and disappear completely for values of thermal mean occupancy exceeding the threshold $\frac{1}{2}$. Also destroyed are the sub-Poissonian character of the photon statistics and the well known pairwise oscillations in the photon number distribution. Finally, we have verified that this threshold has the same significance in the case of amplitude-squared squeezing of an SNS.

Appendix. Moments of the quadrature operators

We recall first some basic formulae involving Gauss hypergeometric functions. Thus, the series expansion of a Gauss function is [23]

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad |z| < 1 \tag{A1} $$

where $(a)_n \equiv \Gamma(a+n)/\Gamma(a)$ is Pochhammer's symbol. Notice also the series expansion of the confluent hypergeometric function (Kummer's function)

$$ {}_1F_1(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}. \tag{A2} $$

We have employed in section 2 the relation between a Laguerre polynomial and a confluent hypergeometric function,

$$ L_l^{(\mu)}(x) = \frac{(\mu+1)_l}{l!} \, {}_1F_1(-l;\mu+1;x) \tag{A3} $$

in order to take advantage of the well known integral [23]

$$ \int_0^{\infty} \mathrm{d}t \, \exp(-st) t^{c-1} \, {}_1F_1(a;c;\sigma t) \, {}_1F_1(a';c;\sigma' t) = \Gamma(c) s^{a+a'-c} (s-\sigma)^{-a} (s-\sigma')^{-a'} \times {}_2F_1[a,a';c;\sigma\sigma'(s-\sigma)^{-1}(s-\sigma')^{-1}]. \tag{A4} $$

In the body of the paper we have exploited Humbert's summation formula ([24] p 6, equation (2)),

$$ \exp(-xz) \, {}_1F_1(a;c;z) = \sum_{n=0}^{\infty} \frac{(-xz)^n}{n!} \, {}_2F_1\left(-n,a;c;\frac{1}{x}\right). \tag{A5} $$

In the following we mention a useful power series involving Gauss hypergeometric polynomials (see [23] section 2.5.1)

$$ \sum_{n=0}^{\infty} \frac{(-\sigma)_n}{n!} (t)^n \, {}_2F_1(-n,b;c;z) = (1-t)^{\sigma} \, {}_2F_1\left(-\sigma,b;c;\frac{tz}{t-1}\right). \tag{A6} $$

and its special case $\sigma = -c$,

$$ \sum_{n=0}^{\infty} \frac{(c)_n}{n!} t^n \, {}_2F_1(-n,b;c;z) = (1-t)^{b-c}(1-t+tz)^{-b}. \tag{A7} $$

Now, the way is paved for deriving equation (4.7). By inverting the order of the second and third summations in equation (4.6) we arrive at

$$ \langle (X_1)^N \rangle_{TSN} = N! \left(\frac{1}{2}\right)^{N/2} \sum_{k=0}^{N/2} \sum_{p=0}^{N-2k} \frac{1}{k!p!} (\cosh r)^{N-2k-2p} (2\bar{n}-1)^p \times {}_2F_1\left(-\frac{N}{2}+k+p,-M;1;2\right) \mathcal{S}_1(k,p). \tag{A8} $$

In equation (A8) we have separated the sum

$$ \mathcal{S}_1(k,p) = \sum_{q=0}^{N-2k-2p} \frac{\exp[-\mathrm{i}(N-2k-2p-2q)\phi/2]}{q!(\frac{N}{2}-k-p-q)!} (\tanh r)^{\frac{N}{2}-k-p-q} \times {}_2F_1\left(-\frac{N}{2}+k+p,-q;\frac{N}{2}-k-p-q+1;(\tanh r)^2\right). \tag{A9} $$

The summation in equation (A9) can be performed with equation (A7), after using the connection between the hypergeometric functions of arguments z and $1-z$. ([23] section 2.9, equation (33)). We obtain

$$\mathcal{S}_1(k,\,p) = \frac{1}{(N/2-k-p)!}\left[\frac{\sinh(2r)\cos\phi+\cosh(2r)}{(\cosh r)^2}\right]^{\frac{N}{2}-k-p}. \qquad \text{(A10)}$$

We insert the result (A10) in equation (A8) and perform the two remaining summations by repeated use of equation (A6). A little algebra leads us to the simple and compact result (4.7).

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